

# $q$ -Deformation of $W(2, 2)$ Lie algebra associated with quantum groups<sup>1</sup>

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**Abstract.** An explicit realization of the  $W(2, 2)$  Lie algebra is presented using the famous bosonic and fermionic oscillators in physics, which is then used to construct the  $q$ -deformation of this Lie algebra. Furthermore, the quantum group structures on the  $q$ -deformation of this Lie algebra are completely determined.

**Key words:**  $W(2, 2)$  Lie algebra,  $q$ -deformation, quantum groups

## §1. Introduction

Since 1990s, there have been intensive explorations of quantized universal enveloping algebras, namely, quantum groups, which were first introduced independently by Drinfeld [7, 8] and Jimbo [16, 17] around 1985 in order for them to construct solutions to the quantum Yang-Baxter equations. Since then quantum groups are found to have numerous applications in various areas ranging from statistical physics via symplectic geometry and knot theory to modular representations of reductive algebraic groups. For this reason, the interests in quantum groups, quantum deformations of Lie algebras as well as Lie bialgebras have been growing in the physical and mathematical literatures, especially those of Cartan type and Block type, which are closely related to the Virasoro algebra and the  $W$ -infinity algebra  $\mathcal{W}_{1+\infty}$  (e.g., [20, 28, 29, 30, 31, 32, 33, 34, 35]). In particular, the  $q$ -deformed Virasoro algebra,  $q$ -deformed oscillator and  $q$ -deformed Heisenberg algebra have been investigated in a number of papers (see e.g. [1, 3, 4, 14, 26, 25]). Among these kinds of algebras, the  $q$ -deformation of the Virasoro algebra has been most intensively considered [1, 10, 13, 19, 23, 24], which can be viewed as a typical example of the physical application of the quantum group. In addition, two-parameter deformation of Lie algebras has also been considered by some authors (see e.g. [2, 5]), while the more general quantum Lie algebras have been investigated in [6, 9, 15, 27]. Roughly speaking, quantum Lie algebras in the context of these deformations are universal enveloping algebras deformed by one or more parameter(s) ( $q$ -deformation) and possess structures of Hopf algebras. However, the essential reason for the name “quantum” algebra is that it becomes the conventional Lie algebra in the  $q \rightarrow 1$  limit (classical limit).

The  $W(2, 2)$  Lie algebra was introduced by Zhang-Dong in [36] for the study of classification of vertex operator algebras generated by vectors of weight 2. Later the Harish-Chandra modules of this Lie algebra were investigated in [21], while the classification of irreducible weight modules was discussed in [22]. The derivations, central extensions and automorphism groups of this Lie algebra were determined in [12]. Recently, the Verma modules over the  $W(2, 2)$  Lie algebra was investigated in [18]. A quantum group structure of the  $q$ -deformed  $W(2, 2)$  Lie algebra was also given in [11]. Nonetheless, there are still plenty rooms for our reconsideration on this matter, since our definition of the  $q$ -deformation  $\mathcal{W}_q$  (Proposition 2.2) of this Lie algebra with its origin from physics, is rather

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different and thus the quantum group (associated with the deformation) constructed in this paper seems to be new.

The  $W(2, 2)$  Lie algebra (denoted by  $\mathcal{W}$ ) considered in the present paper is an infinite dimensional Lie algebra with basis  $L_n, W_n$  (for  $n \in \mathbb{Z}$ ) and the Lie brackets

$$[L_m, L_n] = (n - m)L_{m+n}, \quad [L_m, W_n] = (n - m)W_{m+n}, \quad [W_m, W_n] = 0, \quad \forall \quad m, n \in \mathbb{Z}. \quad (1.1)$$

One sees that it is different from that defined in [36], since we drop here the central element. But our next aim is to develop the central extensions of the  $q$ -deformation  $\mathcal{W}_q$ .

In this paper, an explicit realization (Lemma 2.1) of the  $W(2, 2)$  Lie algebra defined in (1.1) is given using the famous bosonic and fermionic oscillators in physics ((2.1) and (2.3)). As a result, the  $q$ -deformation  $\mathcal{W}_q$  (Proposition 2.2) of the  $W(2, 2)$  Lie algebra is obtained by exact calculations. Based on this, Hopf algebraic structures of  $\mathcal{W}_q$  are proposed and thus the quantum group of  $\mathcal{W}_q$  are completely determined. It seems to us that the results in our paper may be of some potential use in mathematical physics.

Throughout this paper,  $\mathbb{F}$  denotes a field of characteristic zero,  $\mathbb{F}^*$  the multiplicative group of nonzero element of  $\mathbb{F}$ . All vector spaces and algebras are assumed to be over  $\mathbb{F}$ . Denote by  $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}^*$  the sets of integers, nonnegative, nonzero integers, respectively.

## §2. Realization of $\mathcal{W}$ and its $q$ -Deformation $\mathcal{W}_q$

In this section, we propose a version of realizing the  $W(2, 2)$  Lie algebra defined in (1.1). Based on this, a quantum deformation of this Lie algebra is constructed. One sees that the  $q$ -deformed  $W(2, 2)$  Lie algebra  $\mathcal{W}_q$  here is rather different from that given in [11].

In the oscillator, the bosonic oscillator  $a$  and its hermitian conjugate  $a^+$  obey the commutation relations:

$$[a, a^+] = aa^+ - a^+a = 1, \quad [1, a^+] = [1, a] = 0. \quad (2.1)$$

It follows by induction on  $n$  that  $[a, (a^+)^n] = n(a^+)^{n-1}$  for all  $n \in \mathbb{Z}$ . Then the generators

$$L_n \equiv (a^+)^{n+1}a \quad (2.2)$$

realize the centerless Virasoro Lie algebra with bracket:

$$[L_m, L_n] = (n - m)L_{m+n}, \quad \forall \quad m, n \in \mathbb{Z}.$$

More details can be consulted in [23].

For the realization of  $W(2, 2)$  Lie algebra  $\mathcal{W}$  defined in (1.1), in addition to the bosonic oscillators  $a$  and  $a^+$ , we introduce the fermionic oscillators  $b$  and  $b^+$  with the anticommutators

$$\{b, b^+\} = bb^+ + b^+b = 1; \quad b^2 = (b^+)^2 = 0. \quad (2.3)$$

Moreover, we set  $[a, b] = [a, b^+] = [a^+, b] = [a^+, b^+] = 0$ .

**Lemma 2.1** *With notations above, generators of the form*

$$L_n \equiv (a^+)^{n+1}a; \quad W_n \equiv (a^+)^{n+1}b^+a, \quad \forall n \in \mathbb{Z}, \quad (2.4)$$

*realize the  $W(2,2)$  Lie algebra  $\mathcal{W}$  under the commutator*

$$[A, B] = AB - BA, \quad \forall A, B \in \mathcal{W}.$$

*Proof.* We have to check that  $L_n$  and  $W_n$  defined in (2.4) satisfy the three relations in (1.1). In fact, from  $[a, (a^+)^n] = n(a^+)^{n-1}$  it follows

$$\begin{aligned} [L_n, L_m] &= (a^+)^{n+1}a(a^+)^{m+1}a - (a^+)^{m+1}a(a^+)^{n+1}a \\ &= (a^+)^{n+1} \left( (a^+)^{m+1}a + (m+1)(a^+)^m \right) a \\ &\quad - (a^+)^{m+1} \left( (a^+)^{n+1}a + (n+1)(a^+)^n \right) a \\ &= (m-n)(a^+)^{m+n+1}a = (m-n)L_{m+n}. \end{aligned}$$

Similarly, one can get the other two equations, namely,  $[L_n, W_m] = (m-n)W_{m+n}$  and  $[W_n, W_m] = 0$ , since  $b^+$  commutes with both  $a$  and  $a^+$  and since  $(b^+)^2 = 0$ .  $\square$

Fix a  $q \in \mathbb{F}^*$  such that  $q$  is not a root of unity. Instead of equation (2.1), we assume that

$$[a, a^+]_{(q^{-1}, q)} = 1. \quad (2.5)$$

Here we use the notation:

$$[A, B]_{(\alpha, \beta)} = \alpha AB - \beta BA. \quad (2.6)$$

From (2.5) it follows by induction on  $n$  that

$$[a, (a^+)^n]_{(q^{-n}, q^n)} = [n]_q (a^+)^{n-1} \quad (2.7)$$

for arbitrary  $n$ , where the general notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

is used. It is clear to see that  $[-n]_q = -[n]_q$ . Furthermore, one can also deduce that

$$q^n[m]_q - q^m[n]_q = [m-n]_q, \quad q^{-n}[m]_q + q^m[n]_q = [m+n]_q. \quad (2.8)$$

Now we have the following result:

**Proposition 2.2** *The generators  $L_n$  and  $W_n$  (for  $n \in \mathbb{Z}$ ) defined in (2.4) satisfy the following relations:*

$$[L_n, L_m]_{(q^{n-m}, q^{m-n})} = [m-n]_q L_{m+n}, \quad (2.9)$$

$$[L_n, W_m]_{(q^{n-m}, q^{m-n})} = [m-n]_q W_{m+n}, \quad (2.10)$$

$$[W_n, W_m]_{(q^{n-m}, q^{m-n})} = 0, \quad (2.11)$$

for all  $m, n \in \mathbb{Z}$ .

*Proof.* Obviously, equation (2.9) holds for  $m = n$  since both sides are equal to 0. Assume that  $n \neq m$ . Then it follows from (2.4), (2.6), (2.7) and (2.8) that

$$\begin{aligned}
[L_n, L_m]_{(q^{n-m}, q^{m-n})} &= q^{n-m}(a^+)^{n+1}a(a^+)^{m+1}a - q^{m-n}(a^+)^{m+1}a(a^+)^{n+1}a \\
&= q^{n+1}(a^+)^{n+1}\left(q^{-m-1}a(a^+)^{m+1}\right)a - q^{m+1}(a^+)^{m+1}\left(q^{-n-1}a(a^+)^{n+1}\right)a \\
&= q^{n+1}(a^+)^{n+1}\left(q^{m+1}(a^+)^{m+1}a + [m+1]_q(a^+)^m\right)a \\
&\quad - q^{m+1}(a^+)^{m+1}\left(q^{n+1}(a^+)^{n+1}a + [n+1]_q(a^+)^n\right)a \\
&= \left(q^{n+1}[m+1]_q - q^{m+1}[n+1]_q\right)(a^+)^{m+n+1}a \\
&= [m-n]_q(a^+)^{m+n+1}a = [m-n]_q L_{m+n}.
\end{aligned}$$

Hence equation (2.9) holds for all  $m, n \in \mathbb{Z}$ . Similarly, one can get equations (2.10) and (2.11) using the facts that  $b^+$  commutes with both  $a$  and  $a^+$  and that  $(b^+)^2 = 0$ .  $\square$

Proposition 2.2 says that the algebra with generators  $L_n, W_n$  (for  $n \in \mathbb{Z}$ ) and the relations (2.9)–(2.11) realizes the  $W(2, 2)$  Lie algebra  $\mathcal{W}$  in the  $q \rightarrow 1$  limit. We call it the  $q$ -deformation of the  $W(2, 2)$  Lie algebra  $\mathcal{W}$ , which will be denoted by  $\mathcal{W}_q$  in the sequel.

Furthermore, we can generalize this deformation by setting

$$[a, a^+]_{(q^c, q)} = 1,$$

where  $c \in \mathbb{F}^*$  and  $c \neq 1$ . Then by induction, one has

$$[a, (a^+)^n]_{(q^{nc}, q^n)} = [n]_q^c (a^+)^{n-1},$$

where

$$[n]_q^c = \frac{q^n - q^{nc}}{q - q^c}.$$

Now we can show that the expressions (2.4) of  $L_n$  and  $W_n$  satisfy the following relations:

$$[L_n, L_m]_{(q^{n-m}, q^{c(n-m)})} = -[n-m]_q^c L_{m+n}, \quad (2.12)$$

$$[L_n, W_m]_{(q^{n-m}, q^{c(n-m)})} = -[n-m]_q^c W_{m+n}, \quad (2.13)$$

$$[W_n, W_m]_{(q^{n-m}, q^{c(n-m)})} = 0, \quad (2.14)$$

for all  $m, n \in \mathbb{Z}$ . From these we see the algebra generated by  $L_n$  and  $W_n$  in (2.4) with relations (2.12)–(2.14) is also a realization of the  $W(2, 2)$  Lie algebra in the  $q \rightarrow 1$  limit, which is called the generalized  $q$ -deformation of  $W(2, 2)$  Lie algebra and is denoted by  $\mathcal{W}_q^c$ . Note that one gets  $\mathcal{W}_q$  (see Proposition 2.2) when  $c = -1$ .

### §3. Quantum Group Structures of $\mathcal{W}_q$

In this section, we give a direct construction of the Hopf algebraic structures of the  $q$ -deformed  $W(2, 2)$  Lie algebra  $\mathcal{W}_q$  defined in previous section.

Fix a  $q \in \mathbb{F}^*$  such that  $q$  is not a root of unity. Then  $\mathcal{U}_q$  is defined as the associative algebra (with 1 and over  $\mathbb{F}$ ) with generators  $\mathcal{T}, \mathcal{T}^{-1}, L_n, W_n$  for  $n \in \mathbb{Z}$  and relations:

$$\begin{aligned}
(R1) \quad & \mathcal{T}\mathcal{T}^{-1} = 1 = \mathcal{T}^{-1}\mathcal{T}; \\
(R2) \quad & \mathcal{T}^m L_n = q^{-2(n+1)m} L_n \mathcal{T}^m; \\
(R3) \quad & \mathcal{T}^m W_n = q^{-2(n+1)m} W_n \mathcal{T}^m; \\
(R4) \quad & q^{n-m} L_n L_m - q^{m-n} L_m L_n = [m-n]_q L_{m+n}; \\
(R5) \quad & q^{n-m} L_n W_m - q^{m-n} W_m L_n = [m-n]_q W_{m+n}; \\
(R6) \quad & q^{n-m} W_n W_m - q^{m-n} W_m W_n = 0.
\end{aligned}$$

Before giving the construction of the Hopf algebraic structures on  $\mathcal{U}_q$ , we have to check whether or not these six relations (R1)–(R6) above ensure a nontrivial associative algebra  $\mathcal{U}_q$ . The following proposition gives a positive answer.

**Proposition 3.3** *The associative algebra  $\mathcal{U}_q$  with generators  $\mathcal{T}, \mathcal{T}^{-1}, L_n, W_n$  ( $n \in \mathbb{Z}$ ) and relations (R1)–(R6) is nontrivial.*

*Proof.* Set  $M := \{L_n, M_n, \mathcal{T}, \mathcal{T}^{-1} \mid n \in \mathbb{Z}\}$ . Let  $T(M)$  be the tensor algebra of  $M$ , which is a free associative algebra generated by  $M$ . Then one has

$$T(M) = \bigoplus_{m=0}^{\infty} T(M)_m,$$

where  $T(M)_m = \overbrace{M \otimes \cdots \otimes M}^m = \text{span}\{v_1 \otimes v_2 \otimes \cdots \otimes v_m \mid v_i \in M, i = 1, 2, \dots, m\}$ . In particular,  $T(M)_0 = \mathbb{F}$  and  $T(M)_1 = M$ . The product on  $T(M)$  is naturally defined by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_m)(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_n.$$

Let  $I$  be the two-sided ideal of  $T(M)$  generated by

$$\mathcal{T} \otimes \mathcal{T}^{-1} - \mathcal{T}^{-1} \otimes \mathcal{T}, \quad q^{n-m} W_n \otimes W_m - q^{m-n} W_m \otimes W_n; \quad (3.1)$$

$$\mathcal{T}^m \otimes L_n - q^{-2(n+1)m} L_n \otimes \mathcal{T}^m, \quad \mathcal{T}^m \otimes W_n - q^{-2(n+1)m} W_n \otimes \mathcal{T}^m; \quad (3.2)$$

$$q^{n-m} L_n \otimes L_m - q^{m-n} L_m \otimes L_n, \quad q^{n-m} L_n \otimes W_m - q^{m-n} W_m \otimes L_n, \quad (3.3)$$

for all  $m, n \in \mathbb{Z}$  and where  $\mathcal{T}^{-n} = (\mathcal{T}^{-1})^n$ . Set  $S(M) := T(M)/I$ . It is obvious that  $S(M)$  is also a  $\mathbb{Z}$ -graded associative algebra with a basis

$$\tilde{B} = \{T^d (\mathcal{T}^{-1})^{d'} L_{i_1}^{k_1} \cdots L_{i_m}^{k_m} W_{j_1}^{l_1} \cdots W_{j_n}^{l_n}\}, \quad (3.4)$$

where  $d, d', k_i, l_j, i_p, j_q \in \mathbb{Z}_+$ ,  $(i, p = 1, 2, \dots, m; j, q = 1, 2, \dots, n)$ ;  $i_1 < i_2 < \cdots < i_m, j_1 < j_2 < \cdots < j_n$ . Let  $\tilde{J}$  be another two-sided ideal of  $T(M)$  generated by the elements of form

$$q^{n-m} L_n \otimes L_m - q^{m-n} L_m \otimes L_n - [L_n, L_m]_{(q^{n-m}, q^{m-n})}, \quad (3.5)$$

$$q^{n-m} L_n \otimes W_m - q^{m-n} W_m \otimes L_n - [L_n, W_m]_{(q^{n-m}, q^{m-n})}, \quad (3.6)$$

together with that in (3.1) and (3.2). Then set

$$\widetilde{\mathcal{U}}_q := T(M)/\tilde{J}.$$

Our aim is to show that  $\tilde{B}$  defined in (3.4) is also a basis of  $\widetilde{\mathcal{U}}_q$ . Let

$$\tilde{B}' = \{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m} | v_i \in M, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_m, m \geq 0\}$$

be a subset of  $T(M)$  and let  $U'$  be the subspace of  $T(M)$  spanned by  $\tilde{B}'$ . We now claim that

$$T(M) = U' \oplus \tilde{J}. \quad (3.7)$$

For any  $v \in T(M)$ , we can write  $v$  as  $v = v^{(m)} + v^{(m-1)} + \cdots + v^0$ , where  $v^{(m)} \neq 0$  for some  $m \geq 0$  and where  $v^{(i)} \in T(M)_i$  with  $i = 0, 1, \dots, m$ . We call  $m$  the *degree* of  $v$ . From (3.1), (3.2), (3.5) and (3.6), it follows

$$v_{i_1} \otimes \cdots \otimes \left( v_{i_k} \otimes v_{i_{k+1}} - v_{i_{k+1}} \otimes v_{i_k} - [v_{i_k}, v_{i_{k+1}}] \right) \otimes \cdots \otimes v_{i_m} \in \tilde{J},$$

namely, the difference between  $v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \cdots \otimes v_{i_m}$  and  $av_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \otimes v_{i_k} \otimes \cdots \otimes v_{i_m}$  (for some  $a \in \mathbb{F}^*$ ) is an element in  $\tilde{J}$  and an element with degree less than  $m$ . So by induction on the degree of  $v$  one can obtain that  $T(M) = U' + \tilde{J}$ .

It remains to show that equation (3.7) is a direct sum, which is equivalent to the linear independence of  $\tilde{B}$  in  $\widetilde{\mathcal{U}}_q$ . Suppose that a nonzero linear combination  $v$  of the elements in  $\tilde{B}'$  is in  $\tilde{J}$ . It follows from (3.1), (3.2), (3.5) and (3.6) that the homogeneous component  $v^{(m)}$  of  $v$  with highest degree must lie in  $\ker \pi$  (by comparing (3.3) with (3.5) and (3.6)), where  $\pi : T(M) \rightarrow S(M)$  is the natural  $\mathbb{Z}$ -graded algebraic homomorphism, namely,

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m}) = v_{i_1} v_{i_2} \cdots v_{i_m}.$$

However,  $v^{(m)}$  is a nonzero linear combination of the elements in  $\tilde{B}'$ , it is impossible to appear in  $\ker \pi$ . This contradiction implies  $\tilde{B}$  is a basis of  $\widetilde{\mathcal{U}}_q$ . Since it is clear that  $\mathcal{U}_q \cong \widetilde{\mathcal{U}}_q/J$ , where  $J$  is the two-sided ideal of  $\widetilde{\mathcal{U}}_q$  generated by  $\mathcal{T}\mathcal{T}^{-1} - 1$ , we obtain that  $\mathcal{U}_q$  is a nontrivial associative algebra with basis

$$\tilde{B} = \{T^d L_{i_1}^{k_1} \cdots L_{i_m}^{k_m} W_{j_1}^{l_1} \cdots W_{j_n}^{l_n}\}, \quad (3.8)$$

where  $d \in \mathbb{Z}$ ,  $k_i, l_j, i_p, j_q \in \mathbb{Z}_+$ ,  $(i, p = 1, 2, \dots, m; j, q = 1, 2, \dots, n)$ ;  $i_1 < i_2 < \cdots < i_m$ ,  $j_1 < j_2 < \cdots < j_n$ .  $\square$

With the above proposition in hand, we can safely proceed with the construction of the Hopf algebraic structures on  $\mathcal{U}_q$  now. This will be done by several lemmas below.

**Lemma 3.4** *There is a unique homomorphism of  $\mathbb{F}$ -algebras  $\Delta : \mathcal{U}_q \rightarrow \mathcal{U}_q \times \mathcal{U}_q$  with*

$$\Delta(\mathcal{T}) = \mathcal{T} \otimes \mathcal{T}, \quad (3.9)$$

$$\Delta(\mathcal{T}^{-1}) = \mathcal{T}^{-1} \otimes \mathcal{T}^{-1}, \quad (3.10)$$

$$\Delta(L_n) = L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n, \quad (3.11)$$

$$\Delta(W_n) = W_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes W_n. \quad (3.12)$$

*Proof.* It is clear that  $\Delta(\mathcal{T}^m) = \mathcal{T}^m \otimes \mathcal{T}^m$  for arbitrary  $m \in \mathbb{Z}$ . We have to show that  $\Delta(\mathcal{T})$ ,  $\Delta(\mathcal{T}^{-1})$ ,  $\Delta(L_n)$  and  $\Delta(W_n)$  satisfy the relations (R1)–(R6). This is trivial for (R1). For (R2) and (R3) it follows directly from (3.9)–(3.12). Now look at (R4): we have

$$\begin{aligned} \Delta(L_n)\Delta(L_m) &= (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n)(L_m \otimes \mathcal{T}^m + \mathcal{T}^m \otimes L_m) \\ &= L_n L_m \otimes \mathcal{T}^{m+n} + L_n \mathcal{T}^m \otimes \mathcal{T}^n L_m + \mathcal{T}^n L_m \otimes L_n \mathcal{T}^m + \mathcal{T}^{m+n} \otimes L_n L_m \\ &= L_n L_m \otimes \mathcal{T}^{m+n} + q^{-2(m+1)n} L_n \mathcal{T}^m \otimes L_m \mathcal{T}^n \\ &\quad + \mathcal{T}^{m+n} \otimes L_n L_m + q^{-2(m+1)n} L_m \mathcal{T}^n \otimes L_n \mathcal{T}^m. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \Delta(L_m)\Delta(L_n) &= L_m L_n \otimes \mathcal{T}^{m+n} + q^{-2(n+1)m} L_m \mathcal{T}^n \otimes L_n \mathcal{T}^m \\ &\quad + \mathcal{T}^{m+n} \otimes L_m L_n + q^{-2(n+1)m} L_n \mathcal{T}^m \otimes L_m \mathcal{T}^n. \end{aligned}$$

Then it follows

$$\begin{aligned} &q^{n-m}\Delta(L_n)\Delta(L_m) - q^{m-n}\Delta(L_m)\Delta(L_n) \\ &= (q^{n-m}L_n L_m \otimes \mathcal{T}^{m+n} - q^{m-n}L_m L_n \otimes \mathcal{T}^{m+n}) + (q^{n-m}\mathcal{T}^{m+n} \otimes L_n L_m - q^{m-n}\mathcal{T}^{m+n} \otimes L_m L_n) \\ &= [n-m]_q (L_{m+n} \otimes \mathcal{T}^{m+n} + \mathcal{T}^{m+n} \otimes L_{m+n}) = [n-m]_q \Delta(L_{m+n}). \end{aligned}$$

Hence, (R4) is preserved by  $\Delta$  and so is it for (R5) and (R6), which can be checked by the similar method. That means  $\Delta$  is an algebraic homomorphism. Consequently,  $\mathcal{U}_q$  is a bialgebra.  $\square$

The map  $\Delta$  from Lemma 3.4 is called the *comultiplication* on  $\mathcal{U}_q$ . We say  $\Delta$  is *coassociative*, if it satisfies  $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ .

**Lemma 3.5** *The comultiplication  $\Delta$  on  $\mathcal{U}_q$  is coassociative.*

*Proof.* We simply have to check that all the generators of  $\mathcal{U}_q$  are mapped both ways by  $(1 \otimes \Delta)\Delta$  and  $(\Delta \otimes 1)\Delta$  to the same image, which simply involves straightforward calculations. We shall take  $L_n$  as an example (others can be done similarly).

$$\begin{aligned} (1 \otimes \Delta)\Delta(L_n) &= (1 \otimes \Delta)(L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) = L_n \otimes \Delta(\mathcal{T}^n) + \mathcal{T}^n \otimes \Delta(L_n) \\ &= L_n \otimes (\mathcal{T}^n \otimes \mathcal{T}^n) + \mathcal{T}^n \otimes (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \\ &= (L_n \otimes \mathcal{T}^n) \otimes \mathcal{T}^n + (\mathcal{T}^n \otimes L_n) \otimes \mathcal{T}^n + (\mathcal{T}^n \otimes \mathcal{T}^n) \otimes L_n \\ &= \Delta(L_n) \otimes \mathcal{T}^n + \Delta(\mathcal{T}^n) \otimes L_n = (\Delta \otimes 1)(L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \\ &= (\Delta \otimes 1)\Delta(L_n). \end{aligned} \quad \square$$

**Lemma 3.6** *There is a unique homomorphism of  $\mathbb{F}$ -algebras  $\varepsilon : \mathcal{U}_q \rightarrow \mathbb{F}$  with*

$$\varepsilon(\mathcal{T}) = \varepsilon(\mathcal{T}^{-1}) = 1 \quad \text{and} \quad \varepsilon(L_n) = \varepsilon(W_n) = 0, \quad (3.13)$$

*for all  $n \in \mathbb{Z}$ . Moreover, the following diagrams are commutative*

$$\begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \text{id} \downarrow & & \downarrow 1 \otimes \varepsilon \\ \mathcal{U}_q & \xrightarrow{\pi_1} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array} \quad \begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \text{id} \downarrow & & \downarrow \varepsilon \otimes 1 \\ \mathcal{U}_q & \xrightarrow{\pi_2} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array}$$

*namely,  $(1 \otimes \varepsilon)\Delta = \pi_1 \circ \text{id}$  and  $(\varepsilon \otimes 1)\Delta = \pi_2 \circ \text{id}$ , where  $\pi_1$  (resp.  $\pi_2$ ) denotes the isomorphism  $u \mapsto u \otimes 1$  (resp.  $u \mapsto 1 \otimes u$ ) for any  $u \in \mathcal{U}_q$ .*

*Proof.* It is straightforward to see that  $(\varepsilon(\mathcal{T}), \varepsilon(\mathcal{T}^{-1}), \varepsilon(L_n), \varepsilon(W_n)) = (1, 1, 0, 0)$  satisfy the relations (R1)–(R6). So we have the homomorphism  $\varepsilon$ . For the commutativity of the diagrams, it can be easily checked on the generators.  $\square$

The homomorphism  $\varepsilon$  from Lemma 3.6 is called the *counit* of  $\mathcal{U}_q$ .

**Lemma 3.7** *There is a unique antiautomorphism  $S$  of  $\mathcal{U}_q$  with*

$$S(L_m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m}, \quad (3.14)$$

$$S(W_m) = -\mathcal{T}^{-m} W_m \mathcal{T}^{-m}, \quad (3.15)$$

$$S(\mathcal{T}) = \mathcal{T}^{-1}, \quad S(\mathcal{T}^{-1}) = \mathcal{T}. \quad (3.16)$$

*In addition, one has  $S^2 = \text{id}$ .*

*Proof.* We need to show that  $(S(\mathcal{T}), S(\mathcal{T}^{-1}), S(L_m), S(W_m))$  satisfy the relations (R1)–(R6) in  $\mathcal{U}_q^{opp}$ . Let us denote the multiplication in  $\mathcal{U}_q^{opp}$  by a “ $\cdot$ ” in order to distinguish it from that in  $\mathcal{U}_q$ . Now (R1) is obvious and it is easy to see that  $S(\mathcal{T}^m) = \mathcal{T}^{-m}$  ( $m \in \mathbb{Z}$ ). For (R2) we have

$$\begin{aligned} S(\mathcal{T}^m) \cdot S(L_m) &= S(L_m) S(\mathcal{T}^m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m} \mathcal{T}^{-m} = -\mathcal{T}^{-m} q^{-2(m+1)m} \mathcal{T}^{-m} L_m \mathcal{T}^{-m} \\ &= q^{-2(m+1)m} S(\mathcal{T}^m) S(L_m) = q^{-2(m+1)m} S(L_m) \cdot S(\mathcal{T}^m). \end{aligned}$$

One can check similarly that (R3) is preserved by  $S$ . Now consider (R4): it follows from (R2) that

$$\begin{aligned} q^{n-m} S(L_n) \cdot S(L_m) &= q^{n-m} S(L_m) S(L_n) = q^{n-m} \mathcal{T}^{-m} L_m \mathcal{T}^{-m} \mathcal{T}^{-n} L_n \mathcal{T}^{-n} \\ &= q^{n-m} q^{-2(m+1)n} \mathcal{T}^{-m-n} L_m q^{2(n+1)m} L_n \mathcal{T}^{-m-n} \\ &= q^{m-n} \mathcal{T}^{-m-n} L_m L_n \mathcal{T}^{-m-n}. \end{aligned}$$

Similarly, one has  $q^{m-n} S(L_m) \cdot S(L_n) = q^{n-m} \mathcal{T}^{-m-n} L_n L_m \mathcal{T}^{-m-n}$ . Then we have

$$\begin{aligned} q^{n-m} S(L_n) \cdot S(L_m) - q^{m-n} S(L_m) \cdot S(L_n) &= \mathcal{T}^{-m-n} (q^{m-n} L_m L_n - q^{n-m} L_n L_m) \mathcal{T}^{-m-n} \\ &= -[m-n]_q \mathcal{T}^{-m-n} L_{m+n} \mathcal{T}^{-m-n} \\ &= [m-n]_q S(L_{m+n}), \end{aligned}$$



namely, the map  $S$  preserves (R4). One can similarly check that (R5) and (R6) are also preserved by  $S$ . So there is indeed a homomorphism  $S : \mathcal{U}_q \rightarrow \mathcal{U}_q^{opp}$  or an antihomomorphism  $S : \mathcal{U}_q \rightarrow \mathcal{U}_q$  satisfying (3.14)-(3.16). Now  $S^2$  is an ordinary homomorphism from  $\mathcal{U}_q$  to  $\mathcal{U}_q$ . One can check easily on the generators that  $S^2 = \text{id}$ , which implies that  $S$  is bijective.  $\square$

The map  $S$  in Lemma 3.7 is called the *antipode* of  $\mathcal{U}_q$ . It is clear that the inverse  $S^{-1}$  of  $S$  is an antiautomorphism, which is given by

$$S^{-1}(\mathcal{T}) = \mathcal{T}^{-1}, \quad S^{-1}(L_m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m}, \quad S^{-1}(W_m) = -\mathcal{T}^{-m} W_m \mathcal{T}^{-m}, \quad \forall \quad m \in \mathbb{Z}.$$

**Lemma 3.8** *The following diagrams are commutative*

$$\begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \iota \circ \varepsilon \downarrow & & \downarrow 1 \otimes S \\ \mathcal{U}_q & \xleftarrow{m} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array} \quad \begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \iota \circ \varepsilon \downarrow & & \downarrow S \otimes 1 \\ \mathcal{U}_q & \xleftarrow{m} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array}$$

where  $m : \mathcal{U}_q \otimes \mathcal{U}_q \rightarrow \mathcal{U}_q$  is the multiplication map, namely,  $m(u \otimes u') = uu'$  for all  $u, u' \in \mathcal{U}_q$ , and where  $\iota : \mathbb{F} \rightarrow \mathcal{U}_q$  is the embedding  $\iota(a) = a1$  for all  $a \in \mathbb{F}$ .

*Proof.* Let us restrict ourselves to the left diagram. The map  $f = m \circ (1 \otimes S) \circ \Delta$  acts on generators as follows:

$$\begin{aligned} \mathcal{T} &\mapsto \mathcal{T} \otimes \mathcal{T} \mapsto \mathcal{T} \otimes \mathcal{T}^{-1} \mapsto \mathcal{T} \mathcal{T}^{-1} = 1, \\ \mathcal{T}^{-1} &\mapsto \mathcal{T}^{-1} \otimes \mathcal{T}^{-1} \mapsto \mathcal{T}^{-1} \otimes \mathcal{T} \mapsto \mathcal{T}^{-1} \mathcal{T} = 1, \\ L_n &\mapsto L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n \mapsto L_n \otimes \mathcal{T}^{-n} + \mathcal{T}^n \otimes (-\mathcal{T}^{-n} L_n \mathcal{T}^{-n}) \mapsto L_n \mathcal{T}^{-n} - L_n \mathcal{T}^{-n} = 0, \\ W_n &\mapsto W_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes W_n \mapsto W_n \otimes \mathcal{T}^{-n} + \mathcal{T}^n \otimes (-\mathcal{T}^{-n} W_n \mathcal{T}^{-n}) \mapsto W_n \mathcal{T}^{-n} - W_n \mathcal{T}^{-n} = 0, \end{aligned}$$

as predicted by the diagram.

To conclude the proof we have to check: If  $f(u) = \iota \circ \varepsilon(u)$  and  $f(v) = \iota \circ \varepsilon(v)$  for  $u, v \in \mathcal{U}_q$ , then also  $f(uv) = \iota \circ \varepsilon(uv)$ . That is not obvious, since  $S$  and  $m$  are not ring homomorphisms. We suppose that  $\Delta(u) = \sum_i u_i \otimes u'_i$  and  $\Delta(v) = \sum_i v_i \otimes v'_i$  in  $\mathcal{U}_q \otimes \mathcal{U}_q$ . Then  $f(uv)$  is given by

$$uv \mapsto \sum_{i,j} u_i v_j \otimes u'_i v'_j \mapsto \sum_{i,j} u_i v_j \otimes S(v'_j) S(u'_i) \mapsto \sum_{i,j} u_i v_j S(v'_j) S(u'_i) \mapsto \sum_i u_i f(v) S(u'_i),$$

since

$$f(v) = m \circ (1 \otimes S) \circ \Delta(v) = m \circ (1 \otimes S)(\sum_j v_j \otimes v'_j) = m(\sum_j v_j \otimes S(v'_j)) = \sum_j v_j S(v'_j).$$

We assume that  $f(v) = \iota \circ \varepsilon(v)$ , so this element is a scalar multiple of 1 and thus central in  $\mathcal{U}_q$ . Therefore

$$f(uv) = \sum_i u_i S(u'_i) f(v) = f(u) f(v) = \iota \circ \varepsilon(uv). \quad \square$$

In general, an  $\mathbb{F}$ -algebra  $A$  together with algebra homomorphisms  $\Delta : A \rightarrow A \otimes A$  and  $\varepsilon : A \rightarrow \mathbb{F}$  and a linear map  $S : A \rightarrow A$  is called a *Hopf algebra*, if  $\Delta$  is coassociative and if the diagrams

in Lemmas 3.6 and 3.8 (with  $\mathcal{U}_q$  is replaced by  $A$ ) commute. One calls  $\Delta$  the *comultiplication*,  $\varepsilon$  the *counit* and  $S$  the *antipode* of the Hopf algebra. A Hopf algebra  $A$  is called *cocommutative*, if  $P \circ \Delta = \Delta$  with  $P(u \otimes v) = v \otimes u$  for all  $u$  and  $v$  in  $A$ . So the Lemmas 3.4–3.8 say:

**Theorem 3.9**  $(\mathcal{U}_q, \Delta, \varepsilon, S)$  defined by (R1)–(R6) and (3.9)–(3.16) is a Hopf algebra, which is neither cocommutative nor commutative.

**Corollary 3.10** As vector spaces, one has

$$\mathcal{U}_q \cong \mathbb{F}[\mathcal{T}, \mathcal{T}^{-1}] \otimes_{\mathbb{F}} U_q,$$

where  $U_q = U(\mathcal{W}_q)$  is the enveloping algebra of  $\mathcal{W}_q$  generated by  $L_n$  and  $W_n$  ( $n \in \mathbb{Z}$ ) with relations (2.9)–(2.11).

**Corollary 3.11**

$$\Delta(\mathcal{T}^r) = \mathcal{T}^r \otimes \mathcal{T}^r, \quad S(\mathcal{T}^r) = \mathcal{T}^{-r}, \quad \forall \quad r \in \mathbb{Z}; \quad (3.17)$$

$$\Delta(L_n^r) = \sum_{i=0}^r \binom{r}{i} L_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} L_n^i, \quad S(L_n^r) = (-1)^r \mathcal{T}^{-rn} L_n^r \mathcal{T}^{-rn}, \quad \forall \quad r \in \mathbb{Z}_+; \quad (3.18)$$

$$\Delta(W_n^r) = \sum_{i=0}^r \binom{r}{i} W_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} W_n^i, \quad S(W_n^r) = (-1)^r \mathcal{T}^{-rn} W_n^r \mathcal{T}^{-rn}, \quad \forall \quad r \in \mathbb{Z}_+; \quad (3.19)$$

for any  $n \in \mathbb{Z}$ .

*Proof.* Equations in (3.17) are easily obtained from (3.9) and (3.16). One sees that the formulas in (3.18) holds trivially for  $r = 0$ , that is,  $\Delta(1) = 1 \otimes 1$  and  $S(1) = 1$ . Using definitions in Lemmas 3.4 and 3.7, one sees that (3.18) holds for  $r = 1$ . Here are then the inductive steps:

$$\begin{aligned} \Delta(L_n^{r+1}) &= (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \left( \sum_{i=0}^r \binom{r}{i} L_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} L_n^i \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i + \mathcal{T}^n L_n^{r-i} \mathcal{T}^{in} \otimes L_n \mathcal{T}^{(r-i)n} L_n^i \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i + L_n^{r-i} \mathcal{T}^{(i+1)n} \otimes \mathcal{T}^{(r-i)n} L_n^{i+1} \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \right) + \sum_{i=1}^{r+1} \binom{r}{i-1} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \right) \\ &= \sum_{i=0}^{r+1} \left( \binom{r}{i} + \binom{r}{i-1} \right) L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \\ &= \sum_{i=0}^{r+1} \binom{r+1}{i} L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i, \end{aligned}$$

and

$$\begin{aligned} S(L_n^{r+1}) &= (-1)^r \mathcal{T}^{-rn} L_n^r \mathcal{T}^{-rn} (-\mathcal{T}^{-n} L_n \mathcal{T}^{-n}) = (-1)^{r+1} \mathcal{T}^{-rn} (L_n^r \mathcal{T}^{-n}) (\mathcal{T}^{-rn} L_n) \mathcal{T}^{-n} \\ &= (-1)^{r+1} \mathcal{T}^{-rn} (q^{-2rn(n+1)} \mathcal{T}^{-n} L_n^r) (q^{2rn(n+1)} L_n \mathcal{T}^{-rn}) \mathcal{T}^{-n} \\ &= (-1)^{r+1} \mathcal{T}^{-(r+1)n} L_n^{r+1} \mathcal{T}^{-(r+1)n}. \end{aligned}$$

Hence equations in (3.18) hold by induction. Equations in (3.19) can be similarly proved.  $\square$

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# $q$ -Deformation of $W(2, 2)$ Lie algebra associated with quantum groups<sup>1</sup>

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**Abstract.** An explicit realization of the  $W(2, 2)$  Lie algebra is presented using the famous bosonic and fermionic oscillators in physics, which is then used to construct the  $q$ -deformation of this Lie algebra. Furthermore, the quantum group structures on the  $q$ -deformation of this Lie algebra are completely determined.

**Key words:**  $W(2, 2)$  Lie algebra,  $q$ -deformation, quantum groups

## §1. Introduction

Since 1990s, there have been intensive explorations of quantized universal enveloping algebras, namely, quantum groups, which were first introduced independently by Drinfeld [7, 8] and Jimbo [16, 17] around 1985 in order for them to construct solutions to the quantum Yang-Baxter equations. Since then quantum groups are found to have numerous applications in various areas ranging from statistical physics via symplectic geometry and knot theory to modular representations of reductive algebraic groups. For this reason, the interests in quantum groups, quantum deformations of Lie algebras as well as Lie bialgebras have been growing in the physical and mathematical literatures, especially those of Cartan type and Block type, which are closely related to the Virasoro algebra and the  $W$ -infinity algebra  $\mathcal{W}_{1+\infty}$  (e.g., [20, 26, 29, 30, 31, 32, 33, 34, 35, 36]). In particular, the  $q$ -deformed Virasoro algebra,  $q$ -deformed oscillator and  $q$ -deformed Heisenberg algebra have been investigated in a number of papers (see e.g. [1, 3, 4, 14, 27, 25]). Among these kinds of algebras, the  $q$ -deformation of the Virasoro algebra has been most intensively considered [1, 10, 13, 19, 23, 24], which can be viewed as a typical example of the physical application of the quantum group. In addition, two-parameter deformation of Lie algebras has also been considered by some authors (see e.g. [2, 5]), while the more general quantum Lie algebras have been investigated in [6, 9, 15, 28]. Roughly speaking, quantum Lie algebras in the context of these deformations are universal enveloping algebras deformed by one or more parameter(s) ( $q$ -deformation) and possess structures of Hopf algebras. However, the essential reason for the name “quantum” algebra is that it becomes the conventional Lie algebra in the  $q \rightarrow 1$  limit (classical limit).

The  $W(2, 2)$  Lie algebra was introduced by Zhang-Dong in [37] for the study of classification of vertex operator algebras generated by vectors of weight 2. Later the Harish-Chandra modules of this Lie algebra were investigated in [21], while the classification of irreducible weight modules was discussed in [22]. The derivations, central extensions and automorphism groups of this Lie algebra were determined in [12]. Recently, the Verma modules over the  $W(2, 2)$  Lie algebra was investigated in [18]. A quantum group structure of the  $q$ -deformed  $W(2, 2)$  Lie algebra was also given in [11]. Nonetheless, there are still plenty rooms for our reconsideration on this matter, since our definition of the  $q$ -deformation  $\mathcal{W}_q$  (Proposition 2.2) of this Lie algebra with its origin from physics, is rather

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different and thus the quantum group (associated with the deformation) constructed in this paper seems to be new.

The  $W(2, 2)$  Lie algebra (denoted by  $\mathcal{W}$ ) considered in the present paper is an infinite dimensional Lie algebra with basis  $L_n, W_n$  (for  $n \in \mathbb{Z}$ ) and the Lie brackets

$$[L_m, L_n] = (n - m)L_{m+n}, \quad [L_m, W_n] = (n - m)W_{m+n}, \quad [W_m, W_n] = 0, \quad \forall \quad m, n \in \mathbb{Z}. \quad (1.1)$$

One sees that it is different from that defined in [37], since we drop here the central element. But our next aim is to develop the central extensions of the  $q$ -deformation  $\mathcal{W}_q$ .

In this paper, an explicit realization (Lemma 2.1) of the  $W(2, 2)$  Lie algebra defined in (1.1) is given using the famous bosonic and fermionic oscillators in physics ((2.1) and (2.3)). As a result, the  $q$ -deformation  $\mathcal{W}_q$  (Proposition 2.2) of the  $W(2, 2)$  Lie algebra is obtained by exact calculations. Based on this, Hopf algebraic structures of  $\mathcal{W}_q$  are proposed and thus the quantum group of  $\mathcal{W}_q$  are completely determined. It seems to us that the results in our paper may be of some potential use in mathematical physics.

Throughout this paper,  $\mathbb{F}$  denotes a field of characteristic zero,  $\mathbb{F}^*$  the multiplicative group of nonzero element of  $\mathbb{F}$ . All vector spaces and algebras are assumed to be over  $\mathbb{F}$ . Denote by  $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}^*$  the sets of integers, nonnegative, nonzero integers, respectively.

## §2. Realization of $\mathcal{W}$ and its $q$ -Deformation $\mathcal{W}_q$

In this section, we propose a version of realizing the  $W(2, 2)$  Lie algebra defined in (1.1). Based on this, a quantum deformation of this Lie algebra is constructed. One sees that the  $q$ -deformed  $W(2, 2)$  Lie algebra  $\mathcal{W}_q$  here is rather different from that given in [11].

In the oscillator, the bosonic oscillator  $a$  and its hermitian conjugate  $a^+$  obey the commutation relations:

$$[a, a^+] = aa^+ - a^+a = 1, \quad [1, a^+] = [1, a] = 0. \quad (2.1)$$

It follows by induction on  $n$  that  $[a, (a^+)^n] = n(a^+)^{n-1}$  for all  $n \in \mathbb{Z}$ . Then the generators

$$L_n \equiv (a^+)^{n+1}a \quad (2.2)$$

realize the centerless Virasoro Lie algebra with bracket:

$$[L_m, L_n] = (n - m)L_{m+n}, \quad \forall \quad m, n \in \mathbb{Z}.$$

More details can be consulted in [23].

For the realization of  $W(2, 2)$  Lie algebra  $\mathcal{W}$  defined in (1.1), in addition to the bosonic oscillators  $a$  and  $a^+$ , we introduce the fermionic oscillators  $b$  and  $b^+$  with the anticommutators

$$\{b, b^+\} = bb^+ + b^+b = 1; \quad b^2 = (b^+)^2 = 0. \quad (2.3)$$

Moreover, we set  $[a, b] = [a, b^+] = [a^+, b] = [a^+, b^+] = 0$ .

**Lemma 2.1** *With notations above, generators of the form*

$$L_n \equiv (a^+)^{n+1}a; \quad W_n \equiv (a^+)^{n+1}b^+a, \quad \forall n \in \mathbb{Z}, \quad (2.4)$$

*realize the  $W(2,2)$  Lie algebra  $\mathcal{W}$  under the commutator*

$$[A, B] = AB - BA, \quad \forall A, B \in \mathcal{W}.$$

*Proof.* We have to check that  $L_n$  and  $W_n$  defined in (2.4) satisfy the three relations in (1.1). In fact, from  $[a, (a^+)^n] = n(a^+)^{n-1}$  it follows

$$\begin{aligned} [L_n, L_m] &= (a^+)^{n+1}a(a^+)^{m+1}a - (a^+)^{m+1}a(a^+)^{n+1}a \\ &= (a^+)^{n+1} \left( (a^+)^{m+1}a + (m+1)(a^+)^m \right) a \\ &\quad - (a^+)^{m+1} \left( (a^+)^{n+1}a + (n+1)(a^+)^n \right) a \\ &= (m-n)(a^+)^{m+n+1}a = (m-n)L_{m+n}. \end{aligned}$$

Similarly, one can get the other two equations, namely,  $[L_n, W_m] = (m-n)W_{m+n}$  and  $[W_n, W_m] = 0$ , since  $b^+$  commutes with both  $a$  and  $a^+$  and since  $(b^+)^2 = 0$ .  $\square$

Fix a  $q \in \mathbb{F}^*$  such that  $q$  is not a root of unity. Instead of equation (2.1), we assume that

$$[a, a^+]_{(q^{-1}, q)} = 1. \quad (2.5)$$

Here we use the notation:

$$[A, B]_{(\alpha, \beta)} = \alpha AB - \beta BA. \quad (2.6)$$

From (2.5) it follows by induction on  $n$  that

$$[a, (a^+)^n]_{(q^{-n}, q^n)} = [n]_q (a^+)^{n-1} \quad (2.7)$$

for arbitrary  $n$ , where the general notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

is used. It is clear to see that  $[-n]_q = -[n]_q$ . Furthermore, one can also deduce that

$$q^n[m]_q - q^m[n]_q = [m-n]_q, \quad q^{-n}[m]_q + q^m[n]_q = [m+n]_q. \quad (2.8)$$

Now we have the following result:

**Proposition 2.2** *The generators  $L_n$  and  $W_n$  (for  $n \in \mathbb{Z}$ ) defined in (2.4) satisfy the following relations:*

$$[L_n, L_m]_{(q^{n-m}, q^{m-n})} = [m-n]_q L_{m+n}, \quad (2.9)$$

$$[L_n, W_m]_{(q^{n-m}, q^{m-n})} = [m-n]_q W_{m+n}, \quad (2.10)$$

$$[W_n, W_m]_{(q^{n-m}, q^{m-n})} = 0, \quad (2.11)$$

for all  $m, n \in \mathbb{Z}$ .

*Proof.* Obviously, equation (2.9) holds for  $m = n$  since both sides are equal to 0. Assume that  $n \neq m$ . Then it follows from (2.4), (2.6), (2.7) and (2.8) that

$$\begin{aligned}
[L_n, L_m]_{(q^{n-m}, q^{m-n})} &= q^{n-m}(a^+)^{n+1}a(a^+)^{m+1}a - q^{m-n}(a^+)^{m+1}a(a^+)^{n+1}a \\
&= q^{n+1}(a^+)^{n+1}\left(q^{-m-1}a(a^+)^{m+1}\right)a - q^{m+1}(a^+)^{m+1}\left(q^{-n-1}a(a^+)^{n+1}\right)a \\
&= q^{n+1}(a^+)^{n+1}\left(q^{m+1}(a^+)^{m+1}a + [m+1]_q(a^+)^m\right)a \\
&\quad - q^{m+1}(a^+)^{m+1}\left(q^{n+1}(a^+)^{n+1}a + [n+1]_q(a^+)^n\right)a \\
&= \left(q^{n+1}[m+1]_q - q^{m+1}[n+1]_q\right)(a^+)^{m+n+1}a \\
&= [m-n]_q(a^+)^{m+n+1}a = [m-n]_q L_{m+n}.
\end{aligned}$$

Hence equation (2.9) holds for all  $m, n \in \mathbb{Z}$ . Similarly, one can get equations (2.10) and (2.11) using the facts that  $b^+$  commutes with both  $a$  and  $a^+$  and that  $(b^+)^2 = 0$ .  $\square$

Proposition 2.2 says that the algebra with generators  $L_n, W_n$  (for  $n \in \mathbb{Z}$ ) and the relations (2.9)–(2.11) realizes the  $W(2, 2)$  Lie algebra  $\mathcal{W}$  in the  $q \rightarrow 1$  limit. We call it the  $q$ -deformation of the  $W(2, 2)$  Lie algebra  $\mathcal{W}$ , which will be denoted by  $\mathcal{W}_q$  in the sequel.

Furthermore, we can generalize this deformation by setting

$$[a, a^+]_{(q^c, q)} = 1,$$

where  $c \in \mathbb{F}^*$  and  $c \neq 1$ . Then by induction, one has

$$[a, (a^+)^n]_{(q^{nc}, q^n)} = [n]_q^c (a^+)^{n-1},$$

where

$$[n]_q^c = \frac{q^n - q^{nc}}{q - q^c}.$$

Now we can show that the expressions (2.4) of  $L_n$  and  $W_n$  satisfy the following relations:

$$[L_n, L_m]_{(q^{n-m}, q^{c(n-m)})} = -[n-m]_q^c L_{m+n}, \quad (2.12)$$

$$[L_n, W_m]_{(q^{n-m}, q^{c(n-m)})} = -[n-m]_q^c W_{m+n}, \quad (2.13)$$

$$[W_n, W_m]_{(q^{n-m}, q^{c(n-m)})} = 0, \quad (2.14)$$

for all  $m, n \in \mathbb{Z}$ . From these we see that the algebra generated by  $L_n$  and  $W_n$  in (2.4) with relations (2.12)–(2.14) is also a realization of the  $W(2, 2)$  Lie algebra when  $q \rightarrow 1$ , which is called the generalized  $q$ -deformation of  $W(2, 2)$  Lie algebra and is denoted by  $\mathcal{W}_q^c$ . Note that one gets  $\mathcal{W}_q$  (see Proposition 2.2) when  $c = -1$ .

### §3. Quantum Group Structures of $\mathcal{W}_q$

In this section, we give a direct construction of the Hopf algebraic structures of the  $q$ -deformed  $W(2, 2)$  Lie algebra  $\mathcal{W}_q$  defined in previous section.



Fix a  $q \in \mathbb{F}^*$  such that  $q$  is not a root of unity. Then  $\mathcal{U}_q$  is defined as the associative algebra (with 1 and over  $\mathbb{F}$ ) with generators  $\mathcal{T}, \mathcal{T}^{-1}, L_n, W_n$  for  $n \in \mathbb{Z}$  and relations:

$$\begin{aligned}
(R1) \quad & \mathcal{T}\mathcal{T}^{-1} = 1 = \mathcal{T}^{-1}\mathcal{T}; \\
(R2) \quad & \mathcal{T}^m L_n = q^{-2(n+1)m} L_n \mathcal{T}^m; \\
(R3) \quad & \mathcal{T}^m W_n = q^{-2(n+1)m} W_n \mathcal{T}^m; \\
(R4) \quad & q^{n-m} L_n L_m - q^{m-n} L_m L_n = [m-n]_q L_{m+n}; \\
(R5) \quad & q^{n-m} L_n W_m - q^{m-n} W_m L_n = [m-n]_q W_{m+n}; \\
(R6) \quad & q^{n-m} W_n W_m - q^{m-n} W_m W_n = 0.
\end{aligned}$$

Before giving the construction of the Hopf algebraic structures on  $\mathcal{U}_q$ , we have to check whether or not these six relations (R1)–(R6) above ensure a nontrivial associative algebra  $\mathcal{U}_q$ . The following proposition gives a positive answer.

**Proposition 3.3** *The associative algebra  $\mathcal{U}_q$  with generators  $\mathcal{T}, \mathcal{T}^{-1}, L_n, W_n$  ( $n \in \mathbb{Z}$ ) and relations (R1)–(R6) is nontrivial.*

*Proof.* Set  $M := \{L_n, M_n, \mathcal{T}, \mathcal{T}^{-1} \mid n \in \mathbb{Z}\}$ . Let  $T(M)$  be the tensor algebra of  $M$ , which is a free associative algebra generated by  $M$ . Then one has

$$T(M) = \bigoplus_{m=0}^{\infty} T(M)_m,$$

where  $T(M)_m = \overbrace{M \otimes \cdots \otimes M}^m = \text{span}\{v_1 \otimes v_2 \otimes \cdots \otimes v_m \mid v_i \in M, i = 1, 2, \dots, m\}$ . In particular,  $T(M)_0 = \mathbb{F}$  and  $T(M)_1 = M$ . The product on  $T(M)$  is naturally defined by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_m)(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_m \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_n.$$

Let  $I$  be the two-sided ideal of  $T(M)$  generated by

$$\mathcal{T} \otimes \mathcal{T}^{-1} - \mathcal{T}^{-1} \otimes \mathcal{T}, \quad q^{n-m} W_n \otimes W_m - q^{m-n} W_m \otimes W_n; \quad (3.1)$$

$$\mathcal{T}^m \otimes L_n - q^{-2(n+1)m} L_n \otimes \mathcal{T}^m, \quad \mathcal{T}^m \otimes W_n - q^{-2(n+1)m} W_n \otimes \mathcal{T}^m; \quad (3.2)$$

$$q^{n-m} L_n \otimes L_m - q^{m-n} L_m \otimes L_n, \quad q^{n-m} L_n \otimes W_m - q^{m-n} W_m \otimes L_n, \quad (3.3)$$

for all  $m, n \in \mathbb{Z}$  and where  $\mathcal{T}^{-n} = (\mathcal{T}^{-1})^n$ . Set  $S(M) := T(M)/I$ . It is obvious that  $S(M)$  is also a  $\mathbb{Z}$ -graded associative algebra with a basis

$$\tilde{B} = \{T^d (\mathcal{T}^{-1})^{d'} L_{i_1}^{k_1} \cdots L_{i_m}^{k_m} W_{j_1}^{l_1} \cdots W_{j_n}^{l_n}\}, \quad (3.4)$$

where  $d, d', k_i, l_j, i_p, j_q \in \mathbb{Z}_+$ ,  $(i, p = 1, 2, \dots, m; j, q = 1, 2, \dots, n)$ ;  $i_1 < i_2 < \cdots < i_m, j_1 < j_2 < \cdots < j_n$ . Let  $\tilde{J}$  be another two-sided ideal of  $T(M)$  generated by the elements of form

$$q^{n-m} L_n \otimes L_m - q^{m-n} L_m \otimes L_n - [L_n, L_m]_{(q^{n-m}, q^{m-n})}, \quad (3.5)$$

$$q^{n-m} L_n \otimes W_m - q^{m-n} W_m \otimes L_n - [L_n, W_m]_{(q^{n-m}, q^{m-n})}, \quad (3.6)$$

together with that in (3.1) and (3.2). Then set

$$\widetilde{\mathcal{U}}_q := T(M)/\tilde{J}.$$

Our aim is to show that  $\tilde{B}$  defined in (3.4) is also a basis of  $\widetilde{\mathcal{U}}_q$ . Let

$$\tilde{B}' = \{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m} | v_i \in M, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_m, m \geq 0\}$$

be a subset of  $T(M)$  and let  $U'$  be the subspace of  $T(M)$  spanned by  $\tilde{B}'$ . We now claim that

$$T(M) = U' \oplus \tilde{J}. \quad (3.7)$$

For any  $v \in T(M)$ , we can write  $v$  as  $v = v^{(m)} + v^{(m-1)} + \cdots + v^0$ , where  $v^{(m)} \neq 0$  for some  $m \geq 0$  and where  $v^{(i)} \in T(M)_i$  with  $i = 0, 1, \dots, m$ . We call  $m$  the *degree* of  $v$ . From (3.1), (3.2), (3.5) and (3.6), it follows

$$v_{i_1} \otimes \cdots \otimes \left( v_{i_k} \otimes v_{i_{k+1}} - v_{i_{k+1}} \otimes v_{i_k} - [v_{i_k}, v_{i_{k+1}}] \right) \otimes \cdots \otimes v_{i_m} \in \tilde{J},$$

namely, the difference between  $v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes v_{i_{k+1}} \otimes \cdots \otimes v_{i_m}$  and  $av_{i_1} \otimes \cdots \otimes v_{i_{k+1}} \otimes v_{i_k} \otimes \cdots \otimes v_{i_m}$  (for some  $a \in \mathbb{F}^*$ ) is an element in  $\tilde{J}$  and an element with degree less than  $m$ . So by induction on the degree of  $v$  one can obtain that  $T(M) = U' + \tilde{J}$ .

It remains to show that equation (3.7) is a direct sum, which is equivalent to the linear independence of  $\tilde{B}$  in  $\widetilde{\mathcal{U}}_q$ . Suppose that a nonzero linear combination  $v$  of the elements in  $\tilde{B}'$  is in  $\tilde{J}$ . It follows from (3.1), (3.2), (3.5) and (3.6) that the homogeneous component  $v^{(m)}$  of  $v$  with highest degree must lie in  $\ker \pi$  (by comparing (3.3) with (3.5) and (3.6)), where  $\pi : T(M) \rightarrow S(M)$  is the natural  $\mathbb{Z}$ -graded algebraic homomorphism, namely,

$$\pi(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_m}) = v_{i_1} v_{i_2} \cdots v_{i_m}.$$

However,  $v^{(m)}$  is a nonzero linear combination of the elements in  $\tilde{B}'$ , it is impossible to appear in  $\ker \pi$ . This contradiction implies  $\tilde{B}$  is a basis of  $\widetilde{\mathcal{U}}_q$ . Since it is clear that  $\mathcal{U}_q \cong \widetilde{\mathcal{U}}_q/J$ , where  $J$  is the two-sided ideal of  $\widetilde{\mathcal{U}}_q$  generated by  $\mathcal{T}\mathcal{T}^{-1} - 1$ , we obtain that  $\mathcal{U}_q$  is a nontrivial associative algebra with basis

$$\tilde{B} = \{T^d L_{i_1}^{k_1} \cdots L_{i_m}^{k_m} W_{j_1}^{l_1} \cdots W_{j_n}^{l_n}\}, \quad (3.8)$$

where  $d \in \mathbb{Z}$ ,  $k_i, l_j, i_p, j_q \in \mathbb{Z}_+$ ,  $(i, p = 1, 2, \dots, m; j, q = 1, 2, \dots, n)$ ;  $i_1 < i_2 < \cdots < i_m$ ,  $j_1 < j_2 < \cdots < j_n$ .  $\square$

With the above proposition in hand, we can safely proceed with the construction of the Hopf algebraic structures on  $\mathcal{U}_q$  now. This will be done by several lemmas below.

**Lemma 3.4** *There is a unique homomorphism of  $\mathbb{F}$ -algebras  $\Delta : \mathcal{U}_q \rightarrow \mathcal{U}_q \times \mathcal{U}_q$  with*

$$\Delta(\mathcal{T}) = \mathcal{T} \otimes \mathcal{T}, \quad (3.9)$$

$$\Delta(\mathcal{T}^{-1}) = \mathcal{T}^{-1} \otimes \mathcal{T}^{-1}, \quad (3.10)$$

$$\Delta(L_n) = L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n, \quad (3.11)$$

$$\Delta(W_n) = W_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes W_n. \quad (3.12)$$

*Proof.* It is clear that  $\Delta(\mathcal{T}^m) = \mathcal{T}^m \otimes \mathcal{T}^m$  for arbitrary  $m \in \mathbb{Z}$ . We have to show that  $\Delta(\mathcal{T})$ ,  $\Delta(\mathcal{T}^{-1})$ ,  $\Delta(L_n)$  and  $\Delta(W_n)$  satisfy the relations (R1)–(R6). This is trivial for (R1). For (R2) and (R3) it follows directly from (3.9)–(3.12). Now look at (R4): we have

$$\begin{aligned} \Delta(L_n)\Delta(L_m) &= (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n)(L_m \otimes \mathcal{T}^m + \mathcal{T}^m \otimes L_m) \\ &= L_n L_m \otimes \mathcal{T}^{m+n} + L_n \mathcal{T}^m \otimes \mathcal{T}^n L_m + \mathcal{T}^n L_m \otimes L_n \mathcal{T}^m + \mathcal{T}^{m+n} \otimes L_n L_m \\ &= L_n L_m \otimes \mathcal{T}^{m+n} + q^{-2(m+1)n} L_n \mathcal{T}^m \otimes L_m \mathcal{T}^n \\ &\quad + \mathcal{T}^{m+n} \otimes L_n L_m + q^{-2(m+1)n} L_m \mathcal{T}^n \otimes L_n \mathcal{T}^m. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \Delta(L_m)\Delta(L_n) &= L_m L_n \otimes \mathcal{T}^{m+n} + q^{-2(n+1)m} L_m \mathcal{T}^n \otimes L_n \mathcal{T}^m \\ &\quad + \mathcal{T}^{m+n} \otimes L_m L_n + q^{-2(n+1)m} L_n \mathcal{T}^m \otimes L_m \mathcal{T}^n. \end{aligned}$$

Then it follows

$$\begin{aligned} &q^{n-m}\Delta(L_n)\Delta(L_m) - q^{m-n}\Delta(L_m)\Delta(L_n) \\ &= (q^{n-m}L_n L_m \otimes \mathcal{T}^{m+n} - q^{m-n}L_m L_n \otimes \mathcal{T}^{m+n}) + (q^{n-m}\mathcal{T}^{m+n} \otimes L_n L_m - q^{m-n}\mathcal{T}^{m+n} \otimes L_m L_n) \\ &= [n-m]_q (L_{m+n} \otimes \mathcal{T}^{m+n} + \mathcal{T}^{m+n} \otimes L_{m+n}) = [n-m]_q \Delta(L_{m+n}). \end{aligned}$$

Hence, (R4) is preserved by  $\Delta$  and so is it for (R5) and (R6), which can be checked by the similar method. That means  $\Delta$  is an algebraic homomorphism. Consequently,  $\mathcal{U}_q$  is a bialgebra.  $\square$

The map  $\Delta$  from Lemma 3.4 is called the *comultiplication* on  $\mathcal{U}_q$ . We say  $\Delta$  is *coassociative*, if it satisfies  $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ .

**Lemma 3.5** *The comultiplication  $\Delta$  on  $\mathcal{U}_q$  is coassociative.*

*Proof.* We simply have to check that all the generators of  $\mathcal{U}_q$  are mapped both ways by  $(1 \otimes \Delta)\Delta$  and  $(\Delta \otimes 1)\Delta$  to the same image, which simply involves straightforward calculations. We shall take  $L_n$  as an example (others can be done similarly).

$$\begin{aligned} (1 \otimes \Delta)\Delta(L_n) &= (1 \otimes \Delta)(L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) = L_n \otimes \Delta(\mathcal{T}^n) + \mathcal{T}^n \otimes \Delta(L_n) \\ &= L_n \otimes (\mathcal{T}^n \otimes \mathcal{T}^n) + \mathcal{T}^n \otimes (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \\ &= (L_n \otimes \mathcal{T}^n) \otimes \mathcal{T}^n + (\mathcal{T}^n \otimes L_n) \otimes \mathcal{T}^n + (\mathcal{T}^n \otimes \mathcal{T}^n) \otimes L_n \\ &= \Delta(L_n) \otimes \mathcal{T}^n + \Delta(\mathcal{T}^n) \otimes L_n = (\Delta \otimes 1)(L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \\ &= (\Delta \otimes 1)\Delta(L_n). \end{aligned} \quad \square$$

**Lemma 3.6** *There is a unique homomorphism of  $\mathbb{F}$ -algebras  $\varepsilon : \mathcal{U}_q \rightarrow \mathbb{F}$  with*

$$\varepsilon(\mathcal{T}) = \varepsilon(\mathcal{T}^{-1}) = 1 \quad \text{and} \quad \varepsilon(L_n) = \varepsilon(W_n) = 0, \quad (3.13)$$

*for all  $n \in \mathbb{Z}$ . Moreover, the following diagrams are commutative*

$$\begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \text{id} \downarrow & & \downarrow 1 \otimes \varepsilon \\ \mathcal{U}_q & \xrightarrow{\pi_1} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array} \quad \begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \text{id} \downarrow & & \downarrow \varepsilon \otimes 1 \\ \mathcal{U}_q & \xrightarrow{\pi_2} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array}$$

*namely,  $(1 \otimes \varepsilon)\Delta = \pi_1 \circ \text{id}$  and  $(\varepsilon \otimes 1)\Delta = \pi_2 \circ \text{id}$ , where  $\pi_1$  (resp.  $\pi_2$ ) denotes the isomorphism  $u \mapsto u \otimes 1$  (resp.  $u \mapsto 1 \otimes u$ ) for any  $u \in \mathcal{U}_q$ .*

*Proof.* It is straightforward to see that  $(\varepsilon(\mathcal{T}), \varepsilon(\mathcal{T}^{-1}), \varepsilon(L_n), \varepsilon(W_n)) = (1, 1, 0, 0)$  satisfy the relations (R1)–(R6). So we have the homomorphism  $\varepsilon$ . For the commutativity of the diagrams, it can be easily checked on the generators.  $\square$

The homomorphism  $\varepsilon$  from Lemma 3.6 is called the *counit* of  $\mathcal{U}_q$ .

**Lemma 3.7** *There is a unique antiautomorphism  $S$  of  $\mathcal{U}_q$  with*

$$S(L_m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m}, \quad (3.14)$$

$$S(W_m) = -\mathcal{T}^{-m} W_m \mathcal{T}^{-m}, \quad (3.15)$$

$$S(\mathcal{T}) = \mathcal{T}^{-1}, \quad S(\mathcal{T}^{-1}) = \mathcal{T}. \quad (3.16)$$

*In addition, one has  $S^2 = \text{id}$ .*

*Proof.* We need to show that  $(S(\mathcal{T}), S(\mathcal{T}^{-1}), S(L_m), S(W_m))$  satisfy the relations (R1)–(R6) in  $\mathcal{U}_q^{opp}$ . Let us denote the multiplication in  $\mathcal{U}_q^{opp}$  by a “ $\cdot$ ” in order to distinguish it from that in  $\mathcal{U}_q$ . Now (R1) is obvious and it is easy to see that  $S(\mathcal{T}^m) = \mathcal{T}^{-m}$  ( $m \in \mathbb{Z}$ ). For (R2) we have

$$\begin{aligned} S(\mathcal{T}^m) \cdot S(L_m) &= S(L_m) S(\mathcal{T}^m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m} \mathcal{T}^{-m} = -\mathcal{T}^{-m} q^{-2(m+1)m} \mathcal{T}^{-m} L_m \mathcal{T}^{-m} \\ &= q^{-2(m+1)m} S(\mathcal{T}^m) S(L_m) = q^{-2(m+1)m} S(L_m) \cdot S(\mathcal{T}^m). \end{aligned}$$

One can check similarly that (R3) is preserved by  $S$ . Now consider (R4): it follows from (R2) that

$$\begin{aligned} q^{n-m} S(L_n) \cdot S(L_m) &= q^{n-m} S(L_m) S(L_n) = q^{n-m} \mathcal{T}^{-m} L_m \mathcal{T}^{-m} \mathcal{T}^{-n} L_n \mathcal{T}^{-n} \\ &= q^{n-m} q^{-2(m+1)n} \mathcal{T}^{-m-n} L_m q^{2(n+1)m} L_n \mathcal{T}^{-m-n} \\ &= q^{m-n} \mathcal{T}^{-m-n} L_m L_n \mathcal{T}^{-m-n}. \end{aligned}$$

Similarly, one has  $q^{m-n} S(L_m) \cdot S(L_n) = q^{n-m} \mathcal{T}^{-m-n} L_n L_m \mathcal{T}^{-m-n}$ . Then we have

$$\begin{aligned} q^{n-m} S(L_n) \cdot S(L_m) - q^{m-n} S(L_m) \cdot S(L_n) &= \mathcal{T}^{-m-n} (q^{m-n} L_m L_n - q^{n-m} L_n L_m) \mathcal{T}^{-m-n} \\ &= -[m-n]_q \mathcal{T}^{-m-n} L_{m+n} \mathcal{T}^{-m-n} \\ &= [m-n]_q S(L_{m+n}), \end{aligned}$$

namely, the map  $S$  preserves (R4). One can similarly check that (R5) and (R6) are also preserved by  $S$ . So there is indeed a homomorphism  $S : \mathcal{U}_q \rightarrow \mathcal{U}_q^{opp}$  or an antihomomorphism  $S : \mathcal{U}_q \rightarrow \mathcal{U}_q$  satisfying (3.14)-(3.16). Now  $S^2$  is an ordinary homomorphism from  $\mathcal{U}_q$  to  $\mathcal{U}_q$ . One can check easily on the generators that  $S^2 = \text{id}$ , which implies that  $S$  is bijective.  $\square$

The map  $S$  in Lemma 3.7 is called the *antipode* of  $\mathcal{U}_q$ . It is clear that the inverse  $S^{-1}$  of  $S$  is an antiautomorphism, which is given by

$$S^{-1}(\mathcal{T}) = \mathcal{T}^{-1}, \quad S^{-1}(L_m) = -\mathcal{T}^{-m} L_m \mathcal{T}^{-m}, \quad S^{-1}(W_m) = -\mathcal{T}^{-m} W_m \mathcal{T}^{-m}, \quad \forall \quad m \in \mathbb{Z}.$$

**Lemma 3.8** *The following diagrams are commutative*

$$\begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \iota \circ \varepsilon \downarrow & & \downarrow 1 \otimes S \\ \mathcal{U}_q & \xleftarrow{m} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array} \quad \begin{array}{ccc} \mathcal{U}_q & \xrightarrow{\Delta} & \mathcal{U}_q \otimes \mathcal{U}_q \\ \iota \circ \varepsilon \downarrow & & \downarrow S \otimes 1 \\ \mathcal{U}_q & \xleftarrow{m} & \mathcal{U}_q \otimes \mathcal{U}_q \end{array}$$

where  $m : \mathcal{U}_q \otimes \mathcal{U}_q \rightarrow \mathcal{U}_q$  is the multiplication map, namely,  $m(u \otimes u') = uu'$  for all  $u, u' \in \mathcal{U}_q$ , and where  $\iota : \mathbb{F} \rightarrow \mathcal{U}_q$  is the embedding  $\iota(a) = a1$  for all  $a \in \mathbb{F}$ .

*Proof.* Let us restrict ourselves to the left diagram. The map  $f = m \circ (1 \otimes S) \circ \Delta$  acts on generators as follows:

$$\begin{aligned} \mathcal{T} &\mapsto \mathcal{T} \otimes \mathcal{T} \mapsto \mathcal{T} \otimes \mathcal{T}^{-1} \mapsto \mathcal{T} \mathcal{T}^{-1} = 1, \\ \mathcal{T}^{-1} &\mapsto \mathcal{T}^{-1} \otimes \mathcal{T}^{-1} \mapsto \mathcal{T}^{-1} \otimes \mathcal{T} \mapsto \mathcal{T}^{-1} \mathcal{T} = 1, \\ L_n &\mapsto L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n \mapsto L_n \otimes \mathcal{T}^{-n} + \mathcal{T}^n \otimes (-\mathcal{T}^{-n} L_n \mathcal{T}^{-n}) \mapsto L_n \mathcal{T}^{-n} - L_n \mathcal{T}^{-n} = 0, \\ W_n &\mapsto W_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes W_n \mapsto W_n \otimes \mathcal{T}^{-n} + \mathcal{T}^n \otimes (-\mathcal{T}^{-n} W_n \mathcal{T}^{-n}) \mapsto W_n \mathcal{T}^{-n} - W_n \mathcal{T}^{-n} = 0, \end{aligned}$$

as predicted by the diagram.

To conclude the proof we have to check: If  $f(u) = \iota \circ \varepsilon(u)$  and  $f(v) = \iota \circ \varepsilon(v)$  for  $u, v \in \mathcal{U}_q$ , then also  $f(uv) = \iota \circ \varepsilon(uv)$ . That is not obvious, since  $S$  and  $m$  are not ring homomorphisms. We suppose that  $\Delta(u) = \sum_i u_i \otimes u'_i$  and  $\Delta(v) = \sum_i v_i \otimes v'_i$  in  $\mathcal{U}_q \otimes \mathcal{U}_q$ . Then  $f(uv)$  is given by

$$uv \mapsto \sum_{i,j} u_i v_j \otimes u'_i v'_j \mapsto \sum_{i,j} u_i v_j \otimes S(v'_j) S(u'_i) \mapsto \sum_{i,j} u_i v_j S(v'_j) S(u'_i) \mapsto \sum_i u_i f(v) S(u'_i),$$

since

$$f(v) = m \circ (1 \otimes S) \circ \Delta(v) = m \circ (1 \otimes S)(\sum_j v_j \otimes v'_j) = m(\sum_j v_j \otimes S(v'_j)) = \sum_j v_j S(v'_j).$$

We assume that  $f(v) = \iota \circ \varepsilon(v)$ , so this element is a scalar multiple of 1 and thus central in  $\mathcal{U}_q$ . Therefore

$$f(uv) = \sum_i u_i S(u'_i) f(v) = f(u) f(v) = \iota \circ \varepsilon(uv). \quad \square$$

In general, an  $\mathbb{F}$ -algebra  $A$  together with algebra homomorphisms  $\Delta : A \rightarrow A \otimes A$  and  $\varepsilon : A \rightarrow \mathbb{F}$  and a linear map  $S : A \rightarrow A$  is called a *Hopf algebra*, if  $\Delta$  is coassociative and if the diagrams

in Lemmas 3.6 and 3.8 (with  $\mathcal{U}_q$  is replaced by  $A$ ) commute. One calls  $\Delta$  the *comultiplication*,  $\varepsilon$  the *counit* and  $S$  the *antipode* of the Hopf algebra. A Hopf algebra  $A$  is called *cocommutative*, if  $P \circ \Delta = \Delta$  with  $P(u \otimes v) = v \otimes u$  for all  $u$  and  $v$  in  $A$ . So the Lemmas 3.4–3.8 say:

**Theorem 3.9**  $(\mathcal{U}_q, \Delta, \varepsilon, S)$  defined by (R1)–(R6) and (3.9)–(3.16) is a Hopf algebra, which is neither cocommutative nor commutative.

**Corollary 3.10** As vector spaces, one has

$$\mathcal{U}_q \cong \mathbb{F}[\mathcal{T}, \mathcal{T}^{-1}] \otimes_{\mathbb{F}} U_q,$$

where  $U_q = U(\mathcal{W}_q)$  is the enveloping algebra of  $\mathcal{W}_q$  generated by  $L_n$  and  $W_n$  ( $n \in \mathbb{Z}$ ) with relations (2.9)–(2.11).

**Corollary 3.11**

$$\Delta(\mathcal{T}^r) = \mathcal{T}^r \otimes \mathcal{T}^r, \quad S(\mathcal{T}^r) = \mathcal{T}^{-r}, \quad \forall \quad r \in \mathbb{Z}; \quad (3.17)$$

$$\Delta(L_n^r) = \sum_{i=0}^r \binom{r}{i} L_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} L_n^i, \quad S(L_n^r) = (-1)^r \mathcal{T}^{-rn} L_n^r \mathcal{T}^{-rn}, \quad \forall \quad r \in \mathbb{Z}_+; \quad (3.18)$$

$$\Delta(W_n^r) = \sum_{i=0}^r \binom{r}{i} W_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} W_n^i, \quad S(W_n^r) = (-1)^r \mathcal{T}^{-rn} W_n^r \mathcal{T}^{-rn}, \quad \forall \quad r \in \mathbb{Z}_+; \quad (3.19)$$

for any  $n \in \mathbb{Z}$ .

*Proof.* Equations in (3.17) are easily obtained from (3.9) and (3.16). One sees that the formulas in (3.18) holds trivially for  $r = 0$ , that is,  $\Delta(1) = 1 \otimes 1$  and  $S(1) = 1$ . Using definitions in Lemmas 3.4 and 3.7, one sees that (3.18) holds for  $r = 1$ . Here are then the inductive steps:

$$\begin{aligned} \Delta(L_n^{r+1}) &= (L_n \otimes \mathcal{T}^n + \mathcal{T}^n \otimes L_n) \left( \sum_{i=0}^r \binom{r}{i} L_n^{r-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r-i)n} L_n^i \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i + \mathcal{T}^n L_n^{r-i} \mathcal{T}^{in} \otimes L_n \mathcal{T}^{(r-i)n} L_n^i \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i + L_n^{r-i} \mathcal{T}^{(i+1)n} \otimes \mathcal{T}^{(r-i)n} L_n^{i+1} \right) \\ &= \sum_{i=0}^r \binom{r}{i} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \right) + \sum_{i=1}^{r+1} \binom{r}{i-1} \left( L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \right) \\ &= \sum_{i=0}^{r+1} \left( \binom{r}{i} + \binom{r}{i-1} \right) L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i \\ &= \sum_{i=0}^{r+1} \binom{r+1}{i} L_n^{r+1-i} \mathcal{T}^{in} \otimes \mathcal{T}^{(r+1-i)n} L_n^i, \end{aligned}$$

and

$$\begin{aligned} S(L_n^{r+1}) &= (-1)^r \mathcal{T}^{-rn} L_n^r \mathcal{T}^{-rn} (-\mathcal{T}^{-n} L_n \mathcal{T}^{-n}) = (-1)^{r+1} \mathcal{T}^{-rn} (L_n^r \mathcal{T}^{-n}) (\mathcal{T}^{-rn} L_n) \mathcal{T}^{-n} \\ &= (-1)^{r+1} \mathcal{T}^{-rn} (q^{-2rn(n+1)} \mathcal{T}^{-n} L_n^r) (q^{2rn(n+1)} L_n \mathcal{T}^{-rn}) \mathcal{T}^{-n} \\ &= (-1)^{r+1} \mathcal{T}^{-(r+1)n} L_n^{r+1} \mathcal{T}^{-(r+1)n}. \end{aligned}$$

Hence equations in (3.18) hold by induction. Equations in (3.19) can be similarly proved.  $\square$

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